A Theory of Locally Convex Hopf Algevbras Part II. More Duality Results and Examples

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Compactly Generated Spaces

- **Convention**: we only work with Hausdorff topological spaces unless stated otherwise.
- **Notation**: let X be a topological space, $\mathfrak{K}(X)$ denotes the collection of all compact subspaces of X, directed by inclusion.
- A topological space X is **compactly generated (CG)**, or X is a k-space, if it satisfies the any of the following equivalent conditions:
  1. $C \subseteq X$ is closed iff $C \cap K$ is closed in K for every $K \in \mathfrak{K}(X)$;
  2. for any topological space Y and a map $f : X \to Y$, we have f is continuous iff $f \circ t$ is continuous for any continuous map $t : K \to X$ from a compact K.
- For any space X, condition 1 defines a finer topology on X, denoted by $k(X)$, called the k-**ification**. of X.
- $k : \text{HausTop} \to \text{HausCG}$ defines an idempotent functor.

Examples of Compactly Generated Spaces

The following spaces are all compactly generated:

- all first countable spaces; in particular, all metrizable spaces; in particular again, all Polish spaces (which is one of the speaker's original motivation);
- all locally compact spaces;
- all (Hausdorff) inductive limits of compactly generated spaces; in particular, all CW-complexes, e.g. $\mathbb{R}^{(\infty)} = \varinjlim \mathbb{R}^n$.

Remark

Steenrod introduced compactly generated spaces into **algebraic topology**, and they gradually become a standard general assumption in modern treatment of algebraic topology. From this point of view, this is a rather mild assumption.

Topological Groups with Compactly Generated Topology
I. Preparation

- $C(K)$ has AP if K is compact;
- AP is stable under forming reduced projective limit;
- $C(K)\overline{\otimes}_{\varepsilon}C(L) = C(K \times L)$ for compact K and L.
- If X is a k-space, then equipped with the **topology of compact convergence**, we have $C(X) = \varprojlim_{K \in \mathfrak{K}(X)} C(K)$, where the connecting maps are given by restriction.
- $C(X)$ is an F-space if X is a k-space that is σ-compact.
- $\overline{\otimes}_{\varepsilon}$ commutes with reduced projective limits of LCS.
- If both X and Y are k-spaces, and $X \times_k Y = k(X \times Y)$, then

$$C(X)\overline{\otimes}_{\varepsilon}C(Y) = \varprojlim_{(K,L) \in \mathfrak{K}(X) \times \mathfrak{K}(Y)} C(K)\overline{\otimes}_{\varepsilon}C(L)$$

$$= \varprojlim_{(K,L) \in \mathfrak{K}(X) \times \mathfrak{K}(Y)} C(K \times L) = \varprojlim_{M \in \mathfrak{K}(X \times_k Y)} C(M) = C(X \times_k Y).$$

Topological Groups with Compactly Generated Topology II. The Main Result

Theorem (W, 24)

Let G be a topological group with compactly generated topology. Then the group operations of G induces an ε-Hopf algebra structure on $C(G)$. If G is σ-compact, then $C(G)$ is (ε, ι)-polar reflexive.

Sketch of the proof.

Taking the k-fication, the group multiplication $\mu : G \times G \to G$ becomes a continuous map $\mu : G \times_k G \to G$, thus induces a well-defined $\Delta : C(G) \to C(G \times_k G) = C(G)\overline{\otimes}_\varepsilon C(G)$. The other structure maps are also induced from the group operations on G and is much easier, so $C(G)$ becomes an ε-Hopf algebra.
If G is σ-compact, note that $C(G)$ has (AP), then the theorem on polar reflexivity applies. □

The Topological Spectrum and Group-like Elements

Notation: $\chi(H)$ the space of all **continuous** characters of the locally convex algebra H, and $\chi_c(H)$ means $\chi(H)$ equipped with the topology of compact convergence, i.e. as a subspace of H'_c; $\chi^{\mathrm{inv}}(H)$ the involutive continuous characters if H is involutive, and $\chi_c^{\mathrm{inv}}(H)$ the corresponding topological space.

An abstract theorem (W, 24)

The following holds:

- If H is an ε-Hopf algebra of class $(\mathcal{F})$, then $\chi_c(H)$ is a topological group under convolution. If H is furthermore involutive, $\chi_c^{\mathrm{inv}}(H)$ is a closed subgroup of $\chi_c(H)$.
- If H is a π-Hopf algebra, then as a subspace of H, the set of group-like elements $\mathrm{Grp}(H)$ is a topological group with multiplication and topology inherited from H.

A Generalized Gelfand Duality–I. Preparation

Let X be a k-space, $A \subseteq C(X)$ a subalgebra, assumed to be self-adjoint if the scalar field is $\mathbb{C}$.

- We say A is **full**, if $f \in A$ and f invertible in $C(X)$ implies $f^{-1} \in A$.
- **Notation**: $A_{[0,1]} := \{f \in A \mid 0 \leq f \leq 1\}$.
- We say $A_{[0,1]} \subseteq C(X)$ **separates closed and compact sets**, if for any closed $C \subseteq X$ and $K \in \mathfrak{K}(X)$ with $C \cap K = \emptyset$, there exists $f \in A_{[0,1]}$, such that $f(C) = \{0\}$ and $f(K) = \{1\}$.
- Equip A with a new locally convex topology τ, we say (A, τ) is **compactly localized**, if for any continuous seminorm q on (A, τ), there exists $K \in \mathfrak{K}(X)$, such that for all $f \in A$, we have $q(f) = 0$ whenever $f|_K = 0$.
- **Example**: M a smooth manifold, $A = C^\infty(M)$ with τ being the topology of compact convergence on all derivatives, then A is full, $A_{[0,1]}$ separates closed and compact sets, and (A, τ) is compactly localized.

A Generalized Gelfand Duality–II. The result

Theorem (W, 24)

Assume X is a k-space, use the above notation and equip A with a locally convex topology τ such that $(A, \tau) \hookrightarrow C(X)$ is continuous. If A is full and $A_{[0,1]}$ separates closed and compact sets, then the map $X \longrightarrow \chi_c(A)$, $x \mapsto \delta_x$ is a homeomorphism. If (A, τ) is furthermore compactly localized, then this map is a homeomorphism.

- When X is compact, and τ is the topology of uniform convergence, we recover the classical Gelfand duality theorem for unital commutative C^*-algebra.
- O. Aristov has pointed out to the speaker that the case $(A, \tau) = C(X)$ is covered in (N. C. Phillips, 1988).
- As an example, one may recover M as a topological space by using $C^\infty(M)$ for a paracompact smooth manifold M.

Applications to Topological Groups

Theorem (W, 24)

Suppose either of the following hold:

- *G is a Lie group, and $\mathcal{H}_G$ the ε-Hopf algebra $C^\infty(G)$;*
- *G is a topological group with compactly generated topology, $\mathcal{H}_G$ the ε-Hopf algebra $C(G)$.*

Then, the map $\delta : G \longrightarrow \chi_c(\mathcal{H}_G)$ is an isomorphism of topological groups. The same holds in the complex case, where we consider $\mathcal{H}_G$ as an ε-Hopf-$*$ algebra and replace $\chi_c(\mathcal{H}_G)$ by $\chi_c^{\mathrm{inv}}(\mathcal{H}_G)$.

- There is no restriction on the "size" of G in the above.
- In general, it is still unknown whether $\chi_c(H)$ is always a topological group.
- This means that our notion is indeed quite reasonable!

The Eymard-Stinespring-Tatsumma Duality

The **Eymard-Stinespring-Tatsumma duality theorem** for locally compact groups also has a counterpart in this setting.

Theorem (W, 24)

If $\mathcal{H}_{\widehat{G}}$ is the π-Hopf algebra given by any of the following:

- *the strong dual of the ε-Hopf algebra $\mathcal{H}_G = C^\infty(G)$ for a second countable Lie group G;*
- *the polar dual of the ε-Hopf algebra $\mathcal{H}_G = C(G)$ for a topological group with compactly generated topology that is σ-compact.*

Then the map $G \longrightarrow \mathrm{Grp}(\mathcal{H}_{\widehat{G}})$, $g \mapsto \delta_g$ is an isomorphism of topological groups.

The Pontryagin Duality

- Let $\mathbb{K}$ be the scalar field, which is $\mathbb{R}$ or $\mathbb{C}$.
- A group-like element a in a locally convex Hopf algebra H is called **involutive**, if $Sa = a^*$.

Theorem (W, 24)

Let G be a locally compact group and $C(G)$ the associated ε-Hopf algebra.

1. *An element $f \in C(G)$ is group-like if and only if $f : G \to \mathbb{K}$ is a continuous (one-dimensional) representation of G.*
2. *In the complex case and consider $C(G)$ as an ε-Hopf-$*$ algebra. An element $f \in C(G)$ is an involutive group-like elements if and only if $f : G \to \mathbb{C}$ is a unitary representation. Moreover, if G is abelian, then $\mathrm{Grp}^{\mathrm{inv}}(C(G))$, when equipped with the subspace topology induced from $C(G)$, is exactly the Pontryagin dual $\widehat{G}$ of G.*

Projective limits of locally convex spaces

- Let $(E_i, p_i)_{i \in I}$ be a projective system of LCS.
- Let $E = \varprojlim E_i$ be the algebraic projective limit, and $p_i : E \longrightarrow E_i$.
- There exists a unique coarsest locally convex topology on E making each p_i continuous, equipped with this topology, E is called the **projective limit** of $(E_i, p_i)_{i \in I}$.
- If each E_i is Hausdorff, then so is E.
- If each E_i is complete, then so is E.
- If $p_i(E)$ is dense in E_i for each i, then we say the projective limit is **reduced**.
- $\overline{\otimes}_\pi$ and $\overline{\otimes}_\varepsilon$ commutes with *reduced* projective limits.

Inductive limits of locally convex spaces

- Let $(E_i)_{i\in I}$ be an inductive system of locally convex spaces.
- Let $E = \varinjlim E_i$ be the algebraic inductive limit, and $u_i : E_i \longrightarrow E$ canonical.
- There exists a unique finest locally convex topology on E making each u_i continuous, and equipped with this topology E is a LCS, called the **inductive limit** of $(E_i)_i$.
- In general, E is *neither* complete *nor* Hausdorff even if all E_i's are.
- If $I = \mathbb{N}$, and the transition maps $E_n \longrightarrow E_{n+1}$ are isomorphism onto its image, then the inductive limit E is called **sequential and strict**.
- Sequential strict inductive limit, by contrast, preserves Hausdorffness as well as completeness.
- $\otimes_\iota$ commutes with all inductive limits.

Projective and Inductive Limits

- One can form the **projective limit** of **arbitrary (reduced)** projective system of π-Hopf algebras.
- One can also form **inductive limit** of a **strict sequential** inductive system of ι-Hopf algebras.

Theorem (W, 24)

Let $(H_n, u_n)_{n\geq 1}$ of a strict inductive system of ι-Hopf(-$*$) algebras of class $(\mathcal{FN})$, and H its strict inductive limit. Then

1. *for each n, the transpose $p_n : (H_{n+1})'_b \longrightarrow (H_n)'_b$ of u_n is a morphism of ε-Hopf(-$*$) algebras and is surjective as a morphism in $\widehat{\mathsf{LCS}}$, giving rise to a reduced projective system $\left((H_n)'_b, p_n\right)$ of ε-Hopf algebras;*
2. *the ι-Hopf(-$*$) algebra H is (ι, ε)-reflexive, and its strong dual is canonically isomorphic to the projective limit $\varprojlim (H_n)'_b$.*

Some Classical Examples of Inductive Limits

- Consider a strictly increasing sequence of second countable compact Lie groups $(G_n)_{n\geq 1}$ with G_n a closed subgroup of G_{n+1}, and $u_n : G_n \hookrightarrow G_{n+1}$ the embedding, $p_n : C^\infty(G_{n+1}) \to C^\infty(G_n)$ the restriction. Then $\big(C^\infty(G_n), p_n\big)$ is a reduced projective system of ε-Hopf algebras.
- Let $H_\infty = \varprojlim(C^\infty(G_n), p_n)$. Then $\chi_c(H_\infty)$ should be the formal strict inductive limit of (G_n).
- $\chi_c(H_\infty)$ is a topological group, and each G_n embeds canonically into $\chi_c(H_\infty)$ as closed subgroups, with (G_n) strictly increases to $\chi_c(H_\infty)$ as sets. In particular, $\chi_c(H_\infty)$ is not locally compact (it fails Baire's category theorem).
- Moreover, H_∞ is (π, ι)-reflexive.
- In the above, one may take $G_n = S_n$, O_n or U_n.

Quantum Group Examples

- Replace G_n in the previous slide by *separable* compact quantum groups.
- $\mathrm{Pol}(G)$ is nuclear since G is separable, so $\otimes_\varepsilon = \otimes_\pi = \otimes_\iota$ for $\mathrm{Pol}(G)$.
- When the CQG G is separable, the strong dual $\mathrm{Pol}(G)'$ is a π-Hopf algebra of class $(\mathcal{FN})$.
- The subgroup condition becomes $p_n : \mathrm{Pol}(G_{n+1}) \to \mathrm{Pol}(G_n)$ being a surjective π-Hopf algebra morphism.
- When $G_n = S_n^+$, then H_∞ should be the function algebra of S_∞^+.
- When $G_n = O_n^+$, then H_∞ should be the function algebra of O_∞^+.
- When $G_n = U_n^+$, then H_∞ should be the function algebra of U_∞^+.
- These topological quantum groups, defined as locally convex Hopf algebras, *seem not to be locally compact*, but nevertheless still admit a *reasonable strong dual*.

Structures of Locally Compact Groups

We now heavily uses the work related to the solution of Hilbert's fifth problem (Gleason, Montgomery & Zippin, Yamabe etc.). Let G be any locally compact group with G/G_0 compact.

- Call a compact normal subgroup K **good**, if G/K is a Lie group.
- Let $\mathfrak{L}(G)$ denote the collection of good subgroups of G. Then $K_1, K_2 \in \mathcal{L}(G) \implies K_1 \cap K_2 \in \mathcal{L}(G)$, and $\cap_{K \in \mathcal{L}(G)} K = \{e\}$.
- Call $f \in C(G)$ **liftably smooth**, if there exists $K \in \mathcal{L}(G)$, such that there exists $f_K \in C^\infty(G/K)$, such that $f = f_K \circ p_K$, where $p_K : G \longrightarrow G/K$ is the canonical projection.
- The space $\mathscr{E}_l(G)$ of all liftably smooth functions is the union of the images of $p_K^* : C^\infty(G/K) \longrightarrow \mathscr{E}_l(G)$.
- A good subgroup inclusion $K_1 \subseteq K_2$ induces a surjective Lie group morphism $G/K_1 \longrightarrow G/K_2$, hence an embedding $C^\infty(G/K_2) \hookrightarrow C^\infty(G/K_1)$, and $\mathscr{E}_l(G)$ can be seen as the inductive limit of these $C^\infty(G/K)$, $K \in \mathfrak{L}(G)$.

A Variant of Bruhat's Regular Functions

Theorem (W, 24)

Assume G is a second countable LCG with compact G/G_0. Then

1. *As a locally convex space, $\mathscr{E}_l(G)$ is complete.*
2. *The group operations induce an ι-Hopf algebra structure on $\mathscr{E}_l(G)$ that is (ι, π)-reflexive.*
3. *The embedding $\mathscr{E}_l(G) \hookrightarrow C(G)$ satisfies the hypothesis of our generalized version of Gelfand duality.*
4. *The map $G \longrightarrow \chi_c(\mathscr{E}_l(G))$, $g \mapsto \delta_g$ is an isomorphism of topological groups.*

By now, our theory can be seen as an alternative approach to the Kac program, and includes many more non-locally compact examples, both classical and quantum, and is able to describe finer Lie group related structures as well.

Thank you

Thank you!